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The Problem with Average Acceleration Problems

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Problems in introductory mechanics concerning the average acceleration of a particle undergoing general rectilinear motion are a source of much confusion to the student. They can be made the source of much light if a clear distinction is made between time averages and space averages. The term "average force" suffers from the same ambiguity and needs similar clarification. This is especially necessary if one wants to express the impulse-momentum and work-energy theorems in terms of an average force.

The equivalent of the following problem has appeared from time to time in various introductory textbooks[1] at the conclusion of the chapter on the kinematics of rectilinear motion with constant acceleration: *A car decelerates from 50 ft/sec to rest over a distance of 100 ft. What was its average acceleration?*

The student flips back through the chapter and finds that the average acceleration of a body is defined as the total change in velocity divided by the total elapsed time

$$\bar{a} = \Delta v / \Delta t. \tag{1}$$

From this definition, he correctly reasons that, even though the acceleration varied in some unknown fashion, the same velocity change could have been achieved in the same time at a constant acceleration equal to the average acceleration. This he quickly sees as the very definition of average acceleration as given by the textbook.

Here the serious student pauses, looks again at the data given (velocity change and distance of travel) and seriously wonders how to obtain the actual elapsed time in order to calculate the average acceleration. However, with unflinching faith, he assures himself that the problem must be solvable by using the formulas developed in the chapter, namely the equations of constant acceleration. He then vaguely concludes that the actual motion must be equivalent to motion at constant acceleration, if the average acceleration is used, and proceeds to calculate

$$2\bar{a}x = v^2 - v_0^2, \tag{2}$$

$$\bar{a} = -2500/200 = -12.5 \text{ ft/sec}^2.$$

A glance at the back of the book confirms this result and the average student hurries on to the next problem (having learned some all-too vague, if not completely erroneous, physics).

However, the rare, inquiring student might be traumatically puzzled as he hears himself say: "Now that I know the average acceleration, I can use its definition to calculate the elapsed time during the actual stopping of the car

$$\Delta t = \Delta v / \bar{a} = 50/12.5 = 4 \text{ sec}.$$

But I know that a car can be braked from 50 ft/sec to rest in a distance of 100 ft in a variety of ways and, in fact, using many different amounts of time! Yet, I can also see that all of these methods would involve the same average acceleration; but that would require the time to be 4 sec—but it could easily be made 5 sec—but . . ."

The paradox is removed by recognizing the vagueness of the term *average acceleration*. In the mind of the student there is, justifiably, no vagueness to this term because the textbook author has precisely defined it as given above in Eq. (1). The trouble is that what the author has defined is the *time* average of the acceleration, and what he is asking for in the problem is the *space* average of the acceleration. In fact, with the data given, the time average of the acceleration is not determined and cannot be calculated; neither is the actual elapsed time determined or calculable.[2] The student was correct in saying that all rectilinear motions going from 50 ft/sec to rest in a distance of 100 ft have a common average acceleration, but it is the *space* averaged acceleration, not the *time* averaged acceleration.

I find this problem extremely useful for bringing home the difference between time and space averages and for emphasizing that in general the adjective *average* must be further specified before it has a precise meaning, Without this realization, students are puzzled in later courses by such concepts as "the average electric field on the surface (or throughout the volume) of a sphere," etc.

A discussion of the general notions of time, distance, surface, and volume averages arises naturally out of a proper treatment of the elementary kinematics problem under consideration here. Furthermore, a discussion of this problem uncovers some interesting statements which can be made about general rectilinear motion in terms of only average accelerations. It is precisely the use and abuse of these general statements which is under analysis here.

By defining the time averaged acceleration as

$$\bar{a}_t = (1/\Delta t) \int a dt = \Delta v/\Delta t, \tag{3}$$

one directly shows that the equation

$$v(t) = v_0 + \bar{a}_t t \tag{4}$$

is valid for any rectilinear motion, so long as $\bar{a}_t$ is the value of the acceleration averaged over the time interval $t=0$ to t. Thus, $\bar{a}_t$ is that constant acceleration which would produce the same velocity change in the same *time* as occurred in the actual motion (but not the same *distance*).

Similarly, by defining the space averaged acceleration as

$$\bar{a}_x = (1/\Delta x) \int a dx = (1/\Delta x) \int v dv, \tag{5}$$

one directly shows that the equation

$$2\bar{a}_x x = v^2 - v_0^2 \tag{6}$$

is valid for any rectilinear motion, so long as $\bar{a}_x$ is the value of the acceleration averaged over the space interval $x=0$ to x. (We of course are defining $x=0$ as the position of the body at $t=0$.) Thus, $\bar{a}_x$ is that constant acceleration which would produce the same velocity change in the same *distance* as occurred in the actual motion (but not the same *time*).

In short, all motions involving the same velocity change in the same time have a common value of $\bar{a}_t$, but not a common value of $\bar{a}_x$ and not a common distance of travel; while all motions involving the same velocity change in the same distance of travel have a common value of $\bar{a}_x$, but not a common value of $\bar{a}_t$ and not a common time of travel.

Using vector notation, the above definitions and results can be generalized to apply to general three-dimensional particle motion, and not just to rectilinear motion:

$$\bar{\mathbf{a}}_t \Delta t = \int \mathbf{a} dt,$$

$$\mathbf{v}(t) = \mathbf{v}_0 + \bar{\mathbf{a}}_t t, \tag{7}$$

$$\bar{\mathbf{a}}_r \cdot \Delta\mathbf{r} = \int \mathbf{a} \cdot d\mathbf{r} = \int \mathbf{v} \cdot d\mathbf{v},$$

$$2\bar{\mathbf{a}}_r \cdot \Delta\mathbf{r} = v^2 - v_0^2. \tag{8}$$

Note that the concept of the dot product is used in the general definition of space averaged acceleration in order to produce a useful concept.[3] It will prove to be especially useful for stating the work-energy theorem in terms of the space average of a varying force (cf. below).

The third equation of constant rectilinear acceleration,

$$x(t) = v_0 t + \tfrac{1}{2} a t^2, \tag{9}$$

is not valid for general rectilinear motion no matter which average value of the acceleration is used for a. The counterpart of this equation for general rectlinear motion involves the use of both $\bar{a}_t$ and $\bar{a}_x$, and can be obtained by eliminating $v(t)$ between Eqs. (4) and (6) to obtain

$$x(t) = (\bar{a}_t/\bar{a}_x) v_0 t + (\bar{a}_t/\bar{a}_x) \tfrac{1}{2} \bar{a}_t t^2, \tag{10}$$

where, as in the other results, the values of $\bar{a}_t$ and $\bar{a}_x$ depend on the particular type of motion involved and are not even constants of the motion, i.e., both average values will in general change as the motion evolves in space and time.

It is a curious result that the displacement equation

$$x(t) = v_0 t + \tfrac{1}{2} a t^2, \tag{11}$$

with $\bar{a}_t$ used for a, is incorrect by an over-all factor $\bar{a}_t/\bar{a}_x$. As a further curiosity, the three-dimensional version of Eq. (10) may be obtained from Eqs. (7) and (8):

$$\bar{\mathbf{a}}_r \cdot \mathbf{\Delta r} = \tfrac{1}{2}\bar{a}_t^2 t^2 + \mathbf{v}_0 \cdot \bar{\mathbf{a}}_t t. \qquad (12)$$

In particle dynamics the term *average force* often suffers from the same ambiguity as does *average acceleration* in kinematics. A similar clarification results from making a clear distinction between the time average and the space average of a varying force. Through Newton's second law, the impulse-momentum and work-energy theorems can then be stated in terms of the time average and the space average, respectively, of a varying force:

$$\bar{\mathbf{F}}_t \Delta t = \int \mathbf{F} dt, \qquad (13)$$

$$\bar{\mathbf{F}}_t \Delta t = \Delta(m\mathbf{v}), \qquad (14)$$

$$\bar{\mathbf{F}}_r \cdot \mathbf{\Delta r} = \int \mathbf{F} \cdot d\mathbf{r}, \qquad (15)$$

$$\bar{\mathbf{F}}_r \cdot \mathbf{\Delta r} = \Delta(\tfrac{1}{2}mv^2). \qquad (16)$$

The relation of these results of dynamics, in terms of average forces, to the previously displayed results of kinematics, stated in terms of average accelerations, is obvious. This suggests that the above discussion of kinematics in terms of average accelerations might be pedagogically useful for removing some of the mystery that seems to surround the student's first encounter with the above theorems of dynamics. Accelerations, velocities, and displacements are intuitively more concrete and more easily visualized (and averaged) than are the notions of force, impulse, momentum, work, and kinetic energy. It seems that much of the very important connection between dynamics and kinematics, probably because it can be so simply and quickly stated, is lost in the usual treatment. Since we find it so useful to speak of average forces in dynamics, why not prepare the ground by giving a more complete and precise treatment of average accelerations in kinematics?

Our above choices for the definitions of space averaged accelerations and forces in the general three-dimensional case require further discussion. These definitions are not the most useful choices for all cases. For instance, in the case of a particle whose speed varies around a closed path, the space averages of the vector acceleration and force, as defined above, become large without limit (i.e., are undefined) as $\mathbf{\Delta r}$ approaches zero.

It is of course the use of the dot product between two vectors on the left-hand side of the defining equations which introduces this difficulty. The dot product is required on the right-hand side in order to generate the very useful concept of kinetic energy, so that the right-hand side is in its most useful form. Since the intuitive notion of "average" expects the average quantity to play the same role in the left-hand expression as the varying quantity plays under the integral sign on the right-hand side, the above choice is a very natural one.

At the price of this symmetry, one might rather define the space average of a varying force through the equation

$$\bar{F}S = \int \mathbf{F} \cdot d\mathbf{r} = \Delta(\tfrac{1}{2}mv^2), \qquad (17)$$

where S is the total scalar distance traveled as measured along the trajectory, and not the final displacement vector, which was used in our previous definition. Note that the space averaged force, as thus defined, is a scalar; in general no direction is associated with it. This latter definition is what most of us have in mind when we use the term "space averaged force" or when we state the work-energy theorem in terms of an "average force."

The important point is that, in the case of vectors, even the specification "space average" is imprecise and still ambiguous. A choice of definition must be made and clearly stated before the term is used by textbook or lecturer. I might suggest that our previous choice, Eq. (15), be termed "displacement average", and that the concept defined by Eq. (17) be termed "distance average," since either concept might be suggested by the general term "space average."

Finally, but perhaps most importantly, I find that even those students who beforehand were bored by a seemingly trivial textbook problem are genuinely fascinated by the above kinematical discussion, all of which is easily visualized—at least for the case of rectilinear motion. If time allows, this problem can be the subject of a lively class discussion, with the instructor playing only a minimum and Socratic type role, until finally the light is turned on: the ambiguity of the word *average*.

[1] Two recent and in many other ways excellent examples are: F. Bueche, *Introduction to Physics for Scientist and Engineers* (McGraw-Hill, New York, 1969), p. 61, probs. 6, 8, 9 (the latter two problems even ask for the elapsed time); and I. M. Freeman, *Physics: Principles and Insights* (McGraw-Hill, New York, 1968), p. 133, prob. 5.12.

[2] This last phrase is very concretely illustrated by drawing velocity vs time curves for several possible motions involving the same velocity change over the same distance (area under the curve), but using different elapsed times. For easy comparison, make one curve the curve of motion with constant acceleration, i.e., a straight line.

[3] It is recognized that the above choice for the definition of the space averaged acceleration of three-dimensional particle motion is a technical definition and does not always coincide with what the layman might have in mind when he uses this term (consider what it says about motion around part or all of a circle at constant or varying speed—what might the layman mean by the term in such cases?). Our choice is motivated solely by its usefulness in producing the general result which follows. This is a pointed lesson in the vagueness of the bare term *average* and in the power of the concept when precisely and usefully defined.

Derivation of a Hamiltonian for a Charged Particle without Recourse to Electromagnetic Potentials

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The Hamiltonian for a charged particle in an electromagnetic field is usually presented in terms of the electromagnetic potentials **A** *and* ϕ. *However for discussions of a large number of atomic and molecular properties, it is more convenient to have available a Hamiltonian which is expressed directly in terms of the more familiar electric field* **E** *and magnetic induction* **B**. *Here it is shown that one can construct a generalized velocity-dependent potential function of* **E** *and* **B**, *which yields the correct expression for the Lorentz force (truncated after the electric quadrupole, magnetic dipole terms). From this generalized potential the corresponding classical Lagrangian and Hamiltonian are obtained directly in terms of* **E** *and* **B**. *The transition to a quantum mechanical* [**E**, **B**]*-dependent Hamiltonian operator is considered, and we list a variety of, static as well as time-dependent, atomic and molecular properties which can be discussed in a unified way using this Hamiltonian as a starting point.*

I. INTRODUCTION

The classical Hamiltonian for a charged particle in an electromagnetic field is conventionally derived by way of the so-called electromagnetic potentials, the vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\phi(\mathbf{r}, t)$, which are functions of position $\mathbf{r}$ and time t. (See, e.g., Goldstein[1] whose notation will be followed closely.) However, it has been known since the work of Goeppert-Mayer,[2] that this $[\mathbf{A}(\mathbf{r}, t), \phi(\mathbf{r}, t)]$-dependent Hamiltonian can be transformed into an equivalent form which contains the electric field vector $\mathbf{E}(\mathbf{r}, t)$ and the magnetic induction $\mathbf{B}(\mathbf{r}, t)$, the latter formulation of the Hamiltonian having more direct appeal to physical intuition (see Refs. 3–6 for more recent discussions of this canonical transformation).

It is the purpose of this communication to demonstrate that one can, in fact, derive the $[\mathbf{E}, \mathbf{B}]$-dependent Hamiltonian directly without any recourse to the electromagnetic potentials $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$. The present derivation starts (Sec. II) from a generalized (velocity-dependent) potential, in terms of $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, based upon an intuitive application of electromagnetism. It is then shown that this generalized potential reproduces the correct expression for the force