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Robert J. Sciamanda

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cal Physics, edited by K. Binder (Springer, New York, 1984), and references therein.

[15]K. F. Freed, "New lattice model for interaction, avoiding problems with controlled length distribution," J. Phys. A **18**, 871–888 (1985); A. M. Nemirovsky, M. G. Bawendi, and K. F. Freed, "Lattice models of polymers solutions: Monomers occupying several lattice sites," J. Chem. Phys. **87**, 7272–7284 (1987); A. I. Pesci and K. F. Freed, "Lattice models of polymers fluids: Monomers occupying several lattice sites. II. Interaction energies," J. Chem. Phys. **90**, 2003–2016 (1989), and references therein.

[16]A. M. Nemirovsky and M. D. Coutinho-Filho, "From dilute to dense self-avoiding walks on hypercubic lattices," J. Stat. Phys. **53**, 1139–1153 (1988); "Packing flexible polymer chains on a lattice," Phys. Rev. A **39**, 3120–3129 (1989).

[17]G. Arfken, *Mathematical Methods for Physicists* (Academic, Orlando, 1985).

[18]H. N. V. Temperley and M. E. Fisher, "Dimer Problem in Statistical Mechanics—An Exact Result," Philos. Mag. XX, 1061–1063 (1961); M. E. Fisher, "Statistical mechanics of dimers on a plane lattice," Phys. Rev. **124**, 1664–1672 (1961); P. W. Kasteleyn, "The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice," Physica **27**, 1209–1225 (1961); J. F. Nagle, "Lattice statistics of hydrogen bonded crystals. I. The residual entropy of ice," J. Math. Phys. 7, 1484–1496 (1966); "Lattice statistics of hydrogen bonded crystals. II. The Slater KDP model and the Rys F- model," J. Math. Phys. **7**, 1492–1496 (1966); E. W. Montroll, in *Applied Combinatorial Mathematics*, edited by E. F. Beckenbach (Wiley, New York, 1964), Chap. 4. W. Kasteleyn, in *Graph Theory and Theoretical Physics*, edited by F. Harary (Academic, New York, 1967), Chap. 2.

[19]P. W. Kasteleyn, "A soluble self-avoiding walk problem," Physica **29**, 1329–1337 (1963); B. Duplantier and F. David, "Exact partition functions and correlation functions of multiple Hamiltonian walks on the Manhattan lattice," J. Stat. Phys. **51**, 327–434 (1988) and references therein.

[20]M. G. Watts, "Application of the method of the Padé approximants to the excluded volume problem," J. Phys. A **8**, 61–66 (1975) and references therein.

[21]M. E. Fisher and D. S. Gaunt, "Ising model and self-avoiding walks on hypercubical lattices and 'high-density' expansions," Phys. Rev. **133**, A224–A239 (1964).

[22]The exponent ν for a SAW in $d=3$ has been calculated using series extrapolation techniques ($\nu = 0.592 \pm 0.002$) by A. J. Guttmann, *Validity of Hyperscaling for the d=3 Ising model*," Phys. Rev. B **33**, 5089–5092 (1986); MC methods ($\nu = 0.5745 \pm 0.0087 \pm 0.0056$) by Ph. de Forcrand *et al.*, "Critical exponents for the self-avoiding random walk in three dimensions," J. Stat. Phys. **49**, 223–234 (1987); and field-theoretic RG methods ($\nu = 0.588 \pm 0.001$) by J. C. Le Guillou and J. Zinn-Justin, "Critical exponents from field theory," Phys. Rev. B **21**, 3976–3998 (1980).

The axial radiation field of a spherical shell section

Robert J. Sciamanda
Department of Physics, Edinboro University of Pennsylvania, Edinboro, Pennsylvania 16444

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Using the Huygens integral an exact, closed-form solution is obtained for the axial radiation field of a spherical shell section source. The calculation is appropriate to an introductory, calculus-based physics course. Applications include ultrasonic inspection transducers and optical diffraction.

The radiation field of a plane, circular source is commonly calculated from the Huygens integral in acoustics courses. Typically a far-field solution is obtained in terms of Bessel functions. The same calculation is performed in optics as a scalar diffraction problem. A related problem is the field of a section of a spherical shell. The axial radiation field of this source admits of an exact Huygens solution in closed form, which is valid at all axial distances. To my knowledge this cannot be said of any other radiation source of finite extent. The calculation is very straightforward and the result involves only a trigonometric function. There is thus much to recommend this problem as an early pedagogical example of radiation field calculations, appropriate even to the calculus-based, introductory physics course.

Such a source has a common application in the concave "focused" piezoelectric transducers used in the ultrasonic inspection of metals, especially welds. In optics this calculation can be applied to the diffraction of a spherical wave by a circular opening in an opaque plane.

Figure 1 illustrates the source as a section of a spherical shell of radius R. The center of curvature of the shell is the point labeled C. The coordinate z of the axial field point P is measured from the shell vertex O. The parameters h and b are measures of the aperture size and are defined in the figure.

Application of the Huygens[1] model is straightforward. Each elemental area, dS, of the shell generates a spherical wave with an amplitude proportional to the elemental area. Each of these waves travels a distance r to the field point P, arriving with its amplitude diminished by a factor of r^{-1}, and its phase advanced by $(\omega t - kr)$.

Because our field point P is on the axis, we may choose the elemental area to be a circular annulus of radius $R \sin\theta$ and thickness $R\,d\theta$, i.e., we take $dS = 2\pi R^2 \sin\theta\, d\theta$. This is permissible because each point on this dS is the same distance r from the axial field point P.

The observed radiation field at P is then calculated as the superposition of the waves from all of the annuli comprising the shell. This calculation is an integral over the surface of the shell. Thus the Huygens integral giving the relative field strength at the axial point P is

$$U(z,t) = \int r^{-1} \exp[i(\omega t - kr)] 2\pi R^2 \sin\theta\, d\theta.$$

This integral[2] is tractable after the law of cosines is used to eliminate θ in favor of r:

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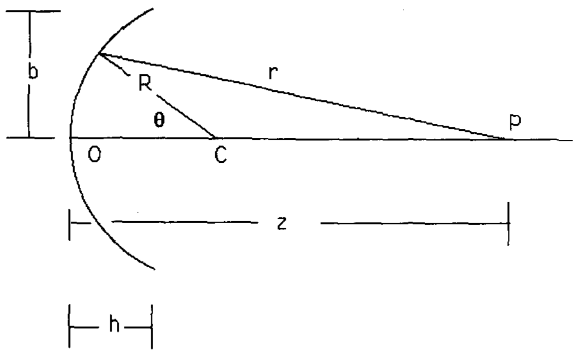

Fig. 1. A spherical shell section radiation source, of radius R. The coordinate of the axial field point P is z, measured from the shell vertex O.

$$r^2 = R^2 + (z-R)^2 + 2R(z-R)\cos\theta$$
$$2r\,dr = -2R(z-R)\sin\theta\,d\theta$$

This transformation of variable yields

$$U(z,t) = -2\pi R\exp(i\omega t) \times (z-R)^{-1}\int_z^{r_{\rm rim}} \exp(-ikr)dr,$$

where $r_{\rm rim}$ is the distance from the rim of the shell to the field point P.

Performing the integration,

$$U(z,t) = -2\pi R\exp(i\omega t)(z-R)^{-1} \times (-ik)^{-1}[\exp(-ikr_{\rm rim}) - \exp(-ikz)].$$

This intermediate result should be explicitly displayed and discussed as a perfect example of Young's unsuccessful radiation theory.[3] That is, except for the $1/(r-R)$ amplitude factor, $U(z,t)$ can be expressed as the superposition of only two waves, one originating at the vertex of the shell (the point O) and the other at its rim. This expression also shows us what to expect of $U(z)$: a two-source interference pattern of maxima and zeros, modulated by the $1/(z-R)$ amplitude factor.

The wave intensity is proportional to U^*U:

$$I(z) = C\,[R\lambda/(z-R)]^2 \sin^2[\pi/\lambda\,(z-r_{\rm rim})],$$

where $\lambda = 2\pi/k$, and all constant factors are absorbed into C.

Referring to Fig. 1, $r_{\rm rim}$ can be evaluated in terms of z, R, and h by eliminating b between the two geometric relations (obvious from the figure):

$$r_{\rm rim}^2 = b^2 + (z-h)^2 \text{ and } R^2 = b^2 + (R-h)^2.$$

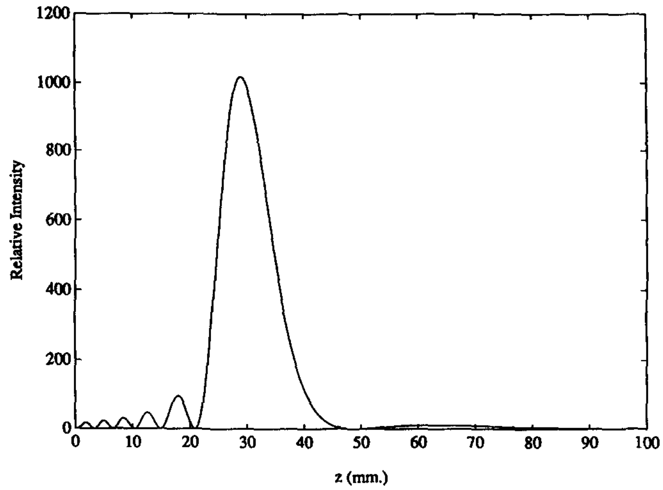

Fig. 2. A plot of the text equation for $I(z)$, the field intensity of an ultrasonic source, with $R = 30$ mm, $h = 10$ mm; and $\lambda = 4$ mm.

These yield

$$r_{\rm rim} = \sqrt{z^2 - 2zh + 2Rh}.$$

Using this result, the intensity becomes

$$I(z) = C\,[R\lambda/(z-R)]^2 \times \sin^2[\pi/\lambda\,(z - \sqrt{z^2 - 2zh + 2Rh})].$$

It is not difficult to show that this final expression is also valid for the convex case, if both h and R are entered as negative quantities.

Figure 2 is a plot of $I(z)$ in which C is taken as 1 and the parameter values are appropriate to an ultrasonic transducer. The two-source interference pattern, modulated by the $(z-R)^{-2}$ factor, is apparent. This same factor concentrates a large fraction of the radiated energy under the principal maximum near the point $z = R$, accounting for the name "focused" transducer. The maxima on the far side $(z > R)$ of the principle maximum are broad and shallow, only the first is (barely) perceptible on this scale.

[1]E. Hecht, *Optics* (Addison-Wesley, Reading, MA, 1990), 2nd ed., pp. 79–81. R. Guenther, *Modern Optics* (Wiley, New York, 1990), pp. 323–329. M. Klein and T. Furtag, *Optics* (Wiley, New York, 1986), pp. 24–25, 2nd ed.

[2]For an introductory course, the complex exponential can be replaced by a sine or cosine. The integral is still easily performed, but the reduction of the result will require some algebraic manipulation using trigonometric identities, exactly as in the treatment of two-source interference.

[3]Thomas Young, "The theory of light and colour," Philos. Trans. R. Soc. London **20**, 12–48 (1802).

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