Mandated energy dissipation—e pluribus unum

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Mandated energy dissipation—*e pluribus unum*

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A single mathematical model describes disparate phenomena involving colliding masses, interacting capacitors, or "dueling" springs. In each case the dissipation of a definite fraction of system energy is mandated when a conservation principle and a final state constraint are imposed. The mathematical "oneness" points not only to the physical similarity but also to a notable physical difference among the various phenomena. An important by-product is an enrichment of the usual treatment of two particle collisions. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

There is a particular textbook problem[1–4] which continues to trouble both physics and engineering students long after they have accepted the mathematical solution which has been thoroughly detailed in this journal.[5,6] This problem is periodically the subject of heated discussions on electronic bulletin boards and "lists," where it has most recently appeared under the subject heading "The Capacitor Conundrum."

The standard treatment of the problem is summarized in Fig. 1. A capacitor is charged by an emf. The emf is then disconnected and the charged capacitor is connected to an uncharged capacitor, with which it shares its charge. The final equilibrium state is determined by the conservation of charge and the requirement of a common final potential difference across the two capacitors. A calculation of the system electrostatic energies in both the "before" and "after" states reveals that a definite fraction of the original system energy has been dissipated; and this fraction depends only on the ratio of the two capacitance values.

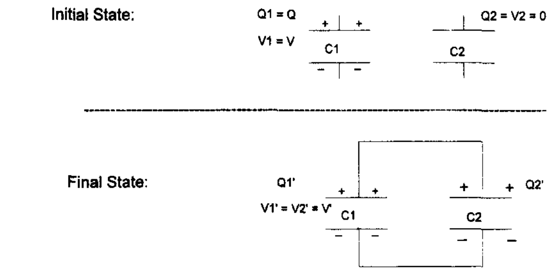
A tractable transient solution is obtained by adding the circuit resistance R to the mathematical model and applying Kirchhoff's loop rule (energy conservation). The solution to the resulting differential equation (with applicable boundary conditions) exactly accounts for the dissipated energy as the Ohmic heat losses incurred during the transient period. As the final equations in Fig. 1 show, this result is independent of the value of the circuit resistance R and is true even in the limit as R is chosen arbitrarily close to zero.

Electromagnetic radiation can also be invoked as the mechanism of energy dissipation;[5] the energy "loss" is the same. The students' "conundrum" seems to be that the amount of energy dissipation is independent of the particulars of the dissipation mechanism. The dissipation of a definite fraction of system energy seems to be an overriding mandate to which particular mechanisms must adjust, regardless of the nature of the mechanism or the value(s) of its parameter(s). Students are wont to speculate (perhaps only subconsciously) that this is the result of some mysterious, unspoken property peculiar to electromagnetic phenomena.

This paper attempts to illuminate this "mystery" by showing that this behavior is not peculiar to interacting capacitors or to electromagnetic phenomena. The same mathematical model also describes some purely mechanical systems, which also operate under a mathematically identical energy dissipation mandate. In the course of unifying these disparate phenomena under a single mathematical model, we will also uncover some subtle physical differences among these systems.

II. MECHANICAL ANALOGUES

Figure 2 describes a mechanical phenomenon which should be quite familiar to the student: a one-dimensional, totally inelastic collision between two "point" masses, as viewed from the frame in which one of the masses is initially at rest. Here, the governing physics consists of the conservation of momentum and the requirement of a common final velocity. It is only a change of notation that distinguishes the mathematical statement of this physics from the governing physics of the interacting capacitors (conservation of charge and a common final potential difference). Furthermore, the same change of notation is all that distinguishes the kinetic energy of the mass system $(1/2)\,\Sigma m_i v_i^2$ from the electrostatic



Conservation of Charge: $Q_1' + Q_2' = Q_1 \;\to\; C_1 V_1' + C_2 V_2' = C_1 V$

Common Final Voltage Constraint: $V_1' = V_2' = V'$

Final Voltage: $V' = C_1 V / (C_1+C_2)$

Initial Energy: $E = (1/2)\, C_1 V^2$ $\to$ **Final Energy:** $E' = E\, C_1 / (C_1 + C_2)$

Fractional Energy Loss: $(E - E') / E = C_2 / (C_1 + C_2)$

Transient behavior:

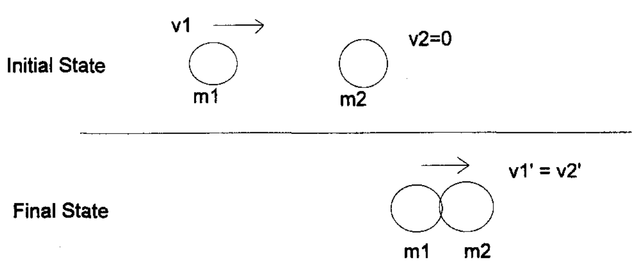
$$V_1(t) + C_1 R\frac{dV_1(t)}{dt} - V_2(t) = 0 \;\to\; V1(t) = \frac{V}{C_1 + C_2}\left(C_1 + C_2 e^{-t\frac{C_1+C_2}{RC_1C_2}}\right)$$

$$i(t) = \frac{V}{R}e^{-t\frac{C_1+C_2}{RC_1C_2}} \;\to\; \int_0^\infty i^2 R\,dt = \frac{V^2}{R}\int_0^\infty e^{-2t\frac{C_1+C_2}{RC_1C_2}} = \frac{V^2 C_1 C_2}{2(C_1 + C_2)}$$

Fig. 1. The standard textbook treatment shows that a charged capacitor is forced to dissipate a definite fraction of its electrostatic energy when it shares its charge with a second capacitor. The transient analysis includes the circuit resistance R and accounts for this energy loss, but the amount of energy dissipated is independent of the value of R.

energy of the capacitor system (1/2) $\Sigma C_i V_i^2$. The "isomorphism" is complete and exact. The mandated energy dissipation of two capacitors forced to a common static voltage is mathematically no more, or less, mysterious than the kinetic energy dissipation mandated by the requirement that two interacting masses achieve a common velocity.

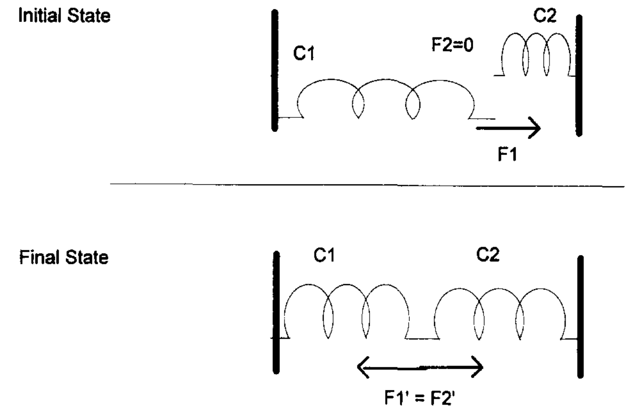
Figure 3 describes a system of two (massless) Hooke's law springs, with compliances C1 and C2, whose behavior will also be modeled by this same mathematics. The far ends



Conservation of Momentum: $P_1' + P_2' = P_1 \;\to\; m_1 v_1' + m_2 v_2' = m_1 v_1$

Common Final Velocity Constraint: $v_1' = v_2'$

Fig. 2. In a totally inelastic collision a definite fraction of the system kinetic energy must be dissipated. The mathematical model is identical to that of the interacting capacitors, except for a change of notation.



Conservation of Elongation: $X_1' + X_2' = X_1 \;\to\; C_1 F_1' + C_2 F_2' = C_1 F_1$

Common Final Force Constraint: $F_1' = F_2'$

Fig. 3. The distal ends of the two (lossy and massless) springs are a fixed horizontal distance apart. The left spring is held stretched, connected to the adjacent end of the other (relaxed) spring, and then let go. A definite fraction of the system elastic energy must be dissipated as the two springs share elongation. This is analogous to the sharing of charge by capacitors; the mathematical model is identical. In the drawing of the initial state the two springs are shown vertically displaced from each other only for clarity.

of the springs are attached to two fixed walls. In the initial state, the left spring is held in a stretched condition by an applied force $F1$ (analogous to an emf charging a capacitor); the spring on the right is relaxed (analogous to an uncharged capacitor). In this "initial state" the two springs just fill the space between the fixed walls. The springs are then made to interact by connecting their adjacent ends together and removing $F1$. The springs will then share elongation (X), just as the capacitors share charge and the colliding masses share momentum. The fixed outer walls guarantee a fixed total elongation X_1+X_2 (for each spring $X=CF$; both X and F are positive for an elongation and negative for a compression). The final state is determined by "conservation of elongation" and the equality of the two spring forces; the system elastic energy is given by (1/2) $\Sigma C_i F_i^2$. Again the isomorphism is exact; whatever mechanisms are involved in achieving the final static state, they must involve the dissipation of a definite fraction of system energy.

The "isomorphism" of the mathematical models of these three phenomena is made explicit in Table I. The first seven rows of Table I should be self-explanatory; the eighth row displays a transient differential equation for each of the phenomena. For the case of the interacting capacitors, this equation was simply reproduced from Fig. 1; a change of notation then produced the differential equations for the colliding masses and for the interacting springs. The physical implementation of these equations (perhaps in terms of "dashpots" between the masses or springs) can be the subject of fruitful class discussion. This will augment the standard treatment of the inelastic collision, which typically suffers from a narrow concern for only the end states and ignores the mechanism or the transient behavior by which these states might be achieved.

The ninth row of Table I exploits the mathematical isomorphisms still further. It is natural to consider viewing the two-mass collision from an alternative inertial frame, through the velocity transformation entered in the ninth row of Table I, under the "inelastic collision" column. That

Table I. Mandated energy dissipation. Three different phenomena, one common mathematical model.

	Inelastic collision of masses	Interaction of capacitors	Interaction of lossy springs
State variables	mass velocities $v_1; v_2$	capacitor voltages $V_1;V_2$	spring forces $F_1;F_2$
Fixed parameters	masses $m_1;m_2$	capacitances $C_1;C_2$	compliances $C_1=1/K_1$; $C_2=1/K_2$
Conserved quantity	momentum $m_1v_1+m_2v_2$	charge $C_1V_1+C_2V_2$	elongation $C_1F_1+C_2F_2$
Initial conditions	$v_1=v$ $v_2=0$	$V_1=V$ $V_2=0$	$F_1=F$ $F_2=0$
Required final condition	common velocity $v_1=v_2$	common potential difference $V_1=V_2$	common spring force $F_1=F_2$
E: System Energy	kinetic energy $(1/2)\Sigma m_iv_i^2$	electrostatic energy $(1/2)\Sigma C_iV_i^2$	elastic potential energy $(1/2)\Sigma C_iF_i^2$
$\Delta E/E$: fractional energy loss	$m_2/(m_1+m_2)$	$C_2/(C_1+C_2)$	$C_2/(C_1+C_2)$
Transient differential equation	$v_1+m_1R\ dv_1/dt-v_2=0$	$V_1+C_1RdV_1/dt-V_2=0$	$F_1+C_1RdF_1/dt-F_2=0$
Transformation to alternative frame	$v_i'=v_i+u$	$V_i'=V_i+U$	$F_i'=F_i+G$

same equation, with the appropriate changes of notation, is then entered in the ninth row under the "capacitor" and "spring" columns. In the capacitor case this transformation adds a fixed voltage U to the (before and after) capacitor voltages; in the spring case the transformation adds a fixed force G to the (before and after) spring forces. The point to be made here is that (as we already know for the collision case) the governing equations are invariant with respect to these transformations, while the state variables will take on transformed before and after values. Lively class discussion can be provoked by considering just how these transformations might be physically achieved (e.g., how about simply pulling apart the exterior walls of the spring system to a new, fixed spacing; or imposing a uniform electric field upon a system of parallel plate capacitors?).

III. A GRAPH IS WORTH A THOUSAND WORDS

It is useful to represent the state of each of these systems by a point in the two-dimensional space spanned by the state variables defined in the first row of Table I. Taking the colliding masses as a paradigm, a plot of the conservation of momentum equation $P_1+P_2=P$ (a constant) is a straight line with a slope of -1, as shown in Fig. 4. Of course, this same plot describes conservation of charge for the capacitor case and conservation of elongation for the spring case. Our before state is on the vertical axis ($P_2=0$). The straight line then shows the only states allowed to the system by the conservation of momentum. If one then draws the curve joining all states having the same system energy (row 6 of Table I) as this initial state, an ellipse results. The eccentricity of the ellipse is completely determined by the mass ratio. For equal masses the ellipse becomes a circle, the case shown in Fig. 4. Curves of lesser/greater system energy are concentric circles (in general, ellipses) of smaller/larger extent. The

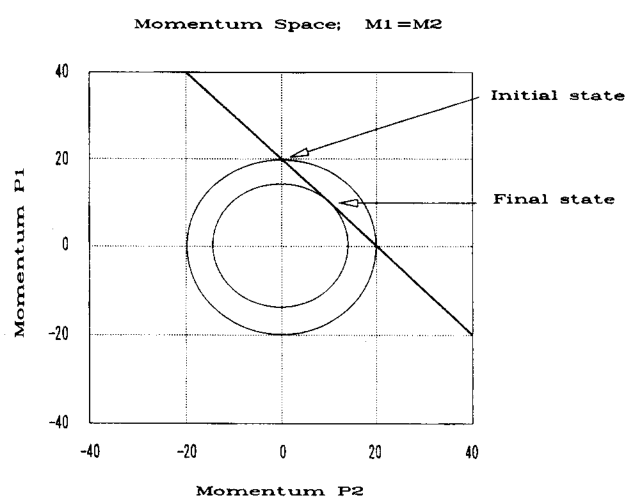


Fig. 4. The momentum space of two identical particles in one-dimensional motion. The straight line, with a slope of -1, connects states of a fixed system momentum. Each circle connects states of a fixed system kinetic energy. The initial and final states of a totally inelastic collision are indicated by the labeled arrows.

course of the system point representing our totally inelastic collision of two equal masses can be seen to be a motion from the initial state $(0,P)$ along the straight line $P_1+P_2=P$ to the final state $(P/2,P/2)$. This clearly demands moving to a circle of lower system energy, the smaller circle drawn in Fig. 4.

This momentum space plot also presents a graphical visualization of the transformation to an alternate inertial frame (row nine of Table I). Under such a transformation, the line of constant momentum simply ''slides'' vertically up or down, maintaining a slope of -1. Of particular interest is the line representing the system as viewed from its center of mass, a line through the origin. This is the line representing zero system momentum and allows the dissipation of all of the system kinetic energy; the circle of final energy states having degenerated into a point at the origin; in the capacitor case this represents a system which begins with equally, but oppositely, charged capacitors. It is most instructive to have students elaborate further upon the properties of these plots as applied to the capacitor and spring systems.

IV. THE ELASTIC POSSIBILITY

It is interesting to apply the plots of Fig. 4 to the perfectly elastic collision of two equal masses in one dimension. Now both system momentum and system kinetic energy must be conserved. Figure 4 shows that there is only one other state where the two conservation plots (the larger circle and the straight line) intersect i.e., on the horizontal axis, where the two particles have simply exchanged velocities. And of course algebra shows that this is indeed the final state for the elastic interaction. But there is a problem. How does the system get from the initial to the final state? Figure 4 clearly shows that there is no continuous path of states connecting these two states which would conserve both momentum and energy throughout the interaction. Indeed, there is no other state in the neighborhood of the initial state which conserves both kinetic energy and momentum.

Since the interaction must obey Newton's laws, it is required that momentum be conserved, and the system must move along the straight line of constant momentum which connects the initial and final states. The first part of this process is from $(0,P)$ to $(P/2,P/2)$, which is the complete path for the inelastic process. During this first step, the elastic system will lose as much kinetic energy as momentum conservation allows, just as does the totally inelastic collision. But, whereas the inelastic process ends here $(P/2,P/2)$, the elastic process keeps traveling along the constant momentum line and terminates at the final state $(P,0)$. During this second part of the elastic process the system gains back all of the kinetic energy which was given up during the first stage.

So the elastic process also demands some mechanism for sinking the particles' kinetic energy, but now this mechanism must be elastic, i.e., it must store this energy in some other form and then return it to the particles as kinetic energy. Just as the inelastic dissipation mechanism can be modeled by a dashpot, so the elastic mechanism can be modeled by a spring; it may, in fact, be a potential energy function. In fact, this treatment brings home the realization of just how our introduction of the potential energy function enabled our first (and purely mechanical) energy conservation theorem, since it is now obvious that kinetic energy cannot be continuously conserved in a one-dimensional, two particle interaction.

V. CONCLUSION

An important illumination is the realization that the one-dimensional interaction (whether elastic or inelastic) of two masses cannot consist solely of an exchange of momentum and kinetic energy between two point entities; there must be present another ''degree of freedom'' in the form of a source and/or sink of energy. Put another way, such an interaction cannot conserve both momentum and kinetic energy throughout the process, so that any conservation of energy theory must invoke transformations of kinetic energy to alternate energy forms. Analogous statements apply to the cases of interacting capacitors and springs. In particular the observation that two capacitors, initially charged to different potentials, cannot come to a common static potential without dissipating electrostatic energy, is isomorphic to the statement that two masses, confined to one dimension and initially having different velocities, cannot come to a common velocity without the dissipation of kinetic energy.

VI. FURTHER SPECULATIONS

A student research project might speculate about extending some of these considerations to collisions in three dimensions. For example, can a three-dimensional collision between two point particles continuously conserve both momentum and kinetic energy?

The students might also speculate upon possible analogs of the perfectly elastic collision for the interacting capacitors and for the interacting springs. It appears that only the mass system has a static (nonoscillatory) elastic solution. The colliding masses can momentarily interact and then go their separate ways with different fixed velocities and no further interaction, while the connected capacitors are forced to a common voltage (and hence a totally inelastic process), and the connected springs must exert equal forces. Among the three systems here defined, only the colliding mass system admits of an elastic mode. In this respect, it is the phenomenon of the colliding masses which stands out as unique, not the phenomenon of the interacting capacitors.

Finally, one might challenge students to conceive a system of two interacting inductors which are described by this same mathematical model, using $E=(1/2)LI^2$ for the energy of an inductance L carrying a current I, and $\Phi=LI$ for the magnetic flux through a circuit (mutual inductance effects would be ignored, just as ''mutual capacitance'' effects were ignored in the original problem). The system would begin with a nonzero current in only one of the inductors (a closed circuit), analogous to beginning with only one of the capacitors charged. By pondering over the first few rows of Table I the students should come to the realization that a gedanken method (don't worry about practicalities) must be contrived so that the inductors will either (a) share this fixed current and attain a common flux or (b) share a fixed amount of flux and attain a common current. They should further realize that this is tantamount to seeking a conservation law, analogous to the conservation of charge/momentum, which (under some contrived constraints) will make this mathematical model apply to interacting inductors. The quest will be very illuminating, even if a physically feasible solution is not uncovered. Any proposed solutions (even if only mathematical and nonphysical) should be added to Table I as a new column. One can then speculate on the possible physical content of the transient differential equation and the transformation to an alternate frame (the entries in rows eight and nine).

Let me add two final observations. The very close parallelism between these three systems, in particular the isomorphism between all three entries for $\Delta E/E$ in Table I, is ultimately due to the fact that all three energy expressions are *quadratic*.

If these energies were instead proportional to V^n, the isomorphism would still hold and the mandated fractional change in energy would now be given by $\Delta E/E = 1-[C_1/(C_1+C_2)]^{n-1}$. Of course, the transient differential equations would also change, since they are simply statements of the conservation of energy.[7] Second, the phenomena considered here demand the presence of a mechanism for the dissipation of system energy. This might be compared, and contrasted, with the phenomenon of pair production, in which a γ particle, no matter how high its energy, cannot turn into an electron–positron pair without losing momentum, so that something else must be present to take up the ''dissipated'' momentum.

[1]This paper is the further development of a paper of the same title delivered orally by the author at a meeting of the Western Pennsylvania Section of the American Association of Physics Teachers, at Carnegie Mellon University, Pittsburgh PA, 29 April 1995 (unpublished).

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[5]C. Cuvaj, ''On Conservation of Energy in Electric Circuits,'' Am. J. Phys. **36**, 909–910 (1968).
[6]R. A. Powell, ''Two-capacitor problem: A more realistic view,'' Am. J. Phys. **47**, 460–462 (1979).

[7]Another approach would be to keep the quadratic energy relation, but to observe that there is also a mandated change in the more generally defined quantity $\Psi=(1/2)\Sigma C_i V_i^n$, leaving its physical meaning open to interpretation (its dimensions would be the product of energy and V^{n-2}). The mandated fractional change in this quantity is given by $\Delta\Psi/\Psi=1-[C_1/(C_1+C_2)]^{n-1}$.

A novel geometry for Rutherford scattering

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We describe a new geometry for a Rutherford scattering experiment which utilizes an annular detector and movable target. The apparatus provides both a good counting rate and simple data reduction. Both the angular dependence and the absolute value of the cross section can be obtained for a reasonable range of scattering angles.

I. INTRODUCTION

Rutherford scattering is potentially an attractive part of an advanced undergraduate laboratory, but it is a technically demanding exercise. The several designs that have appeared in this journal and elsewhere over the years[1–8] represent various compromises intended to balance the needs for accurate cross-section measurements, reasonable count rates, wide angular range, and simple data analysis. In this paper, we describe a new geometry using an annular detector and movable target which leads to a more favorable compromise on some of the important parameters. In particular, we can tightly collimate the incident beam and keep the solid angle of the detector small, allowing for easy data reduction, while still maintaining a high count rate and good accuracy with a modest (1 mCi) source. A secondary benefit is that the solid angle of the detector increases with the scattering angle, partially offsetting the rapid drop in the Rutherford cross section.

II. DESIGN CONSIDERATIONS

As shown schematically in Fig. 1, the sensitive area of the detector is a fixed annulus. The desired scattering angle θ is then set by choosing the distance d between target and detector plane. If N incident alpha particles strike a foil of thickness t containing n scattering centers per unit volume, the average number, dN, of particles scattered into the solid angle $d\Omega$ around Ω is given by

$$dN = Nnt\,\frac{d\sigma}{d\Omega}\,d\Omega \tag{1}$$

The Rutherford cross section for a particle of energy E and charge $2e$ scattering off a massive nucleus of charge Ze is

$$\frac{d\sigma}{d\Omega} = \frac{q^2}{16}\sin^{-4}(\theta/2), \tag{2}$$

where

$$q = 2Ze^2/E. \tag{3}$$

Data analysis is aimed at comparing the observed counting rate $\Delta N/\Delta\Omega$ for the annulus as a function of d with the expected rate according to Eq. (1).

For the simplest analysis, it is reasonable to assume that the beam divergence and beam diameter are small, so that the incident alpha particles form a parallel beam which scatters from a point. Simple geometry and an analytic integration are then sufficient to obtain $\theta(d)$ and $\Delta\Omega(d)$, so that $\Delta N/\Delta\Omega$ can be plotted as a function of $\sin^4(\theta/2)$. Assuming further that $\Delta\Omega$ is small, this ratio should be a good estimate